Sr. No. of Question Paper : 1572
Unique Paper Code : 2352012302
Name of the Paper : DSC-8 : Riemann Integration
Programme : B.Sc. (Hons.) Mathematics (NEP-UGCF 2022)
Semester : III
Duration : 3 Hours

Maximum Marks : 90

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. All questions are compulsory. Attempt any **Three** parts from each question.
3. All questions carry equal marks.

1. (a) Let $f: [-1,1] \to \mathbb{R}$ be defined as follows:

$$f(x) = \begin{cases} 2, & if\ x \in \mathbb{Q} \\ 3, & if\ x \notin \mathbb{Q} \end{cases}$$

Show that f is not integrable on [-1,1].

(b) Let $f: [a,b] \to \mathbb{R}$ be a bounded function. Show that if f is integrable on $[a,b]$, then for each $\varepsilon > 0$, there exists a $\delta > 0$ such that $U(f,P) - L(f,P) < \varepsilon$ for every partition P of $[a,b]$ with mesh$(P) < \delta$.

(c) Let $f(x) = 3x + 2$ over the interval [1,3]. Let P be a partition of [1,3] given by $P = \{1, 3/2, 2, 3\}$. Compute $L(f,P), U(f,P)$ and $U(f,P) - L(f,P)$.

(d) Let $f: [a,b] \to \mathbb{R}$ be a bounded function. Show that if P and Q are any partitions of $[a,b]$, then $L(f,P) \le U(f,Q)$. Hence show that $L(f) \le U(f)$.

(e)

2. (a) Prove that a bounded function f is integrable on $[a,b]$ if and only if there exists a sequence of partitions $(P_n)_{n\in\mathbb{N}}$ of $[a,b]$, satisfying $\lim[U(f,P_n) - L(f,P_n)] = 0$.

(b) Suppose that a function f defined on $[a,b]$ is integrable on $[a,c]$ and $[c,b]$, where $c \in (a,b)$. Prove that f is integrable on $[a,b]$ and that $\int_a^b f = \int_a^c f + \int_c^b f$.

(c) Let $f: [a,b] \to \mathbb{R}$ be a bounded function. Show that if f is Riemann integrable on $[a,b]$, then it is (Darboux) integrable on $[a,b]$, and that the values of the integrals agree.

(d) For $t \in [0,1]$, let $F(t) = \begin{cases} 0 & \text{for } t < 1/3 \\ 1 & \text{for } t \ge 1/3. \end{cases}$

Let $f(x) = x^2$, where $x \in [0,1]$. Show that f is F-integrable and that $\int_0^1 f\, dF = f(1/3)$.

3. (a) Prove that every continuous function on $[a,b]$ is integrable on $[a,b]$.

(b) State and prove the Intermediate Value Theorem for Integrals.